ON THE PROBLEM OF THE EXCITATION OF STANDING WAVES BY A WAVE GENERATOR IN A CHANNEL OF FINITE LENGTH

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PMM Vol.24, No.3, 1960, pp. 551-553

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(Received 1 March 1960)

The problem of the excitation of standing waves by a wave generator in a heavy incompressible fluid in a channel of rectangular cross-section is considered. Principal attention is given to the resonating case in which the frequency of oscillation of the wave generator is close to one of the natural frequencies of oscillation of the fluid in the channel. For the solution of the problem a scheme suggested in [1] is used.

1. Statement of the problem. We shall consider a rectangular channel with vertical walls (Fig. 1). One of the walls (the wave generator), moving in a direction normal to its own plane, is performing the harmonic oscillation

$$x = \varepsilon \sin \omega t$$

Here ω is the frequency, and ϵ is the amplitude of the wave-generator oscillation which is considered to be sufficiently small. In dimensionless variables the problem is reduced to finding a function $\phi(x, z, t)$ harmonic in a region r with the conditions

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=\varepsilon\sin\omega t} = \varepsilon\omega\cos\omega t, \qquad \left.\frac{\partial\varphi}{\partial x}\right|_{x=a} = 0, \qquad \left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = 0$$
$$\frac{\partial\varphi}{\partial t} + \zeta + \frac{1}{2}(\nabla\varphi)^2 = 0, \qquad \frac{\partial\varphi}{\partial x} + (\nabla\varphi\cdot\nabla\zeta) = \frac{\partial\varphi}{\partial z}, \qquad z = 1 + \zeta(x, t) \tag{1.1}$$

Here $z = z(x, t)$ is the equation of the free surface, the depth h is taken as the characteristic dimension, and the potential is referred to $h\sqrt{gh}$.

Let the natural frequencies which correspond to the linear problem be

$$\sigma_n = \sqrt{\lambda_n^* \operatorname{th} \lambda_n^*}, \qquad \lambda_n^* = \frac{n\pi}{a} \qquad (n = 0, 1, 2, \ldots)$$

We shall seek periodic solutions with the period $2\pi/\omega$ which for $\epsilon = 0$ reduce to the trivial solution of the free oscillation problem.

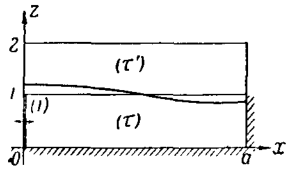

Fig. 1.

2. Oscillations far from resonance ($\omega \neq \sigma_n$). We seek a solution in the form

$$\varphi = \sum_1^\infty \varepsilon^n \varphi_n, \qquad \zeta = \sum_1^\infty \varepsilon^n \zeta_n$$

Considering the oscillations to be small, we can write ($z = 1$)

$$\varphi = \varepsilon\varphi_1 + \varepsilon^2\left(\varphi_2 + \frac{\partial\varphi_1}{\partial z}\zeta_1\right) + \varepsilon^3\left(\varphi_3 + \frac{\partial\varphi_1}{\partial z}\zeta_2 + \frac{\partial\varphi_2}{\partial z}\zeta_1 + \frac{1}{2}\frac{\partial^2\varphi_1}{\partial z^2}\zeta_1^2\right) + \ldots$$

The functions ϕ_i and ζ_i satisfy the conditions

$$\frac{\partial\varphi_1}{\partial t} + \zeta_1 = 0, \quad \frac{\partial\varphi_2}{\partial t} + \zeta_2 = -\frac{\partial}{\partial t}\left[\frac{\partial\varphi_1}{\partial z}\zeta_1\right] - \frac{1}{2}(\nabla\varphi_1)^2, \ldots \text{ for } z = 1 \tag{2.1}$$

$$\frac{\partial\zeta_1}{\partial t} = \frac{\partial\varphi_1}{\partial z}, \quad \frac{\partial\zeta_2}{\partial t} = \frac{\partial\varphi_2}{\partial z} + \frac{\partial^2\varphi_1}{\partial z^2}\zeta_1 - \frac{\partial\varphi_1}{\partial x}\frac{\partial\zeta_1}{\partial x}, \ldots \text{ for } z = 1 \tag{2.2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial\varphi_1}{\partial x} = \omega\cos\omega t, \quad \frac{\partial\varphi_2}{\partial x} = -\frac{\partial^2\varphi_1}{\partial x^2}, \ldots \text{ for } x = 0 \tag{2.3}$$

From the first condition of (2.1) and the first condition of (2.2) we obtain

$$\left(\frac{\partial^2\varphi_1}{\partial t}\right)_{z=1} + \left(\frac{\partial\varphi_1}{\partial z}\right)_{z=1} = 0 \tag{2.4}$$

Let us assume that $\phi_1 = \phi_{11}\cos\omega t + \phi_{12}$, where the function ϕ_{11} is harmonic in the region τ and satisfies the conditions

$$\left(\frac{\partial\varphi_{11}}{\partial x}\right)_{x=0} = \omega \quad \left(\frac{\partial\varphi_{11}}{\partial x}\right)_{x=a} = 0, \quad \left(\frac{\partial\varphi_{11}}{\partial z}\right)_{z=0} = 0$$

The function ϕ_{12} is also harmonic in the region τ and satisfies the conditions

$$\left(\frac{\partial\varphi_{12}}{\partial x}\right)_{x=0} = 0, \quad \left(\frac{\partial\varphi_{12}}{\partial x}\right)_{x=a} = 0, \quad \left(\frac{\partial\varphi_{12}}{\partial z}\right)_{z=0} = 0$$

In order to construct the function ϕ_{11}, we shall add the region τ' to the region r (Fig. 1) so that the condition

$$\left(\frac{\partial\varphi_{11}}{\partial x}\right)_{x=0} = \delta(z)\,\omega, \qquad \delta(z) = \begin{cases} 1 & (0 \leqslant z \leqslant 1) \\ -1 & (1 < z \leqslant 2) \end{cases}$$

is fulfilled.

Let us assume

$$\varphi_{11} = \sum_{1}^{\infty} A_n \varphi_{11n}, \qquad \varphi_{11n} = \operatorname{ch}\frac{n\pi(x-a)}{2}\cos\frac{\varkappa\pi z}{2}$$

Expanding $\delta(z)$ into a Fourier series in terms of $(n\pi z)/2$, we find that

$$A_n = \frac{8\omega}{n^2\pi^2}\sin\frac{n\pi}{2}\operatorname{csh}\frac{n\pi a}{2}$$

From (2.4) we have for ϕ_{12}

$$\left(\frac{\partial^2\varphi_{12}}{\partial t^2}\right)_{z=1} + \left(\frac{\partial\varphi_{12}}{\partial z}\right)_{z=1} = \left(\frac{\partial\varphi_{11}}{\partial z}\right)_{z=1}\cos\omega t$$

Let us assume

$$\varphi_{12} = \sum_{0}^{\infty} f_{1k}(t)\,\varphi_k^* \quad . \quad \left(\varphi_k^* = \sqrt{\frac{2}{a}}\operatorname{ch}\frac{k\pi z}{a}\operatorname{sch}\frac{k\pi}{a}\cos\frac{k\pi x}{a}\right)$$

Here ϕ_k^* is the normalized eigenfunction of the linear problem. Expanding $\operatorname{ch}[1/2\, n\pi(x-a)]$ into a cosine series, we obtain the equation

$$f_{1k}'' + \sigma_k^2 f_{1k} = \frac{4\omega}{\sqrt{2a}}\sum_{n=1}^{\infty}\frac{\sin^2\frac{n\pi}{2}\cos k\pi}{\left(\frac{n\pi}{2}\right)^2 + \left(\frac{k\pi}{a}\right)^2}\cos\omega t = C_k\cos\omega t \tag{2.5}$$

Consequently,

$$f_{1k} = \frac{C_k}{\sigma_k^2 - \omega^2}\cos\omega t$$

$$\varphi_1 = \varepsilon\varphi_{11} + \varepsilon\sum_k \frac{C_k}{\sigma_k^2 - \omega^2}\varphi_k^*\cos\omega t, \quad \zeta_1 = -\left(\frac{\partial\varphi_1}{\partial z}\right)_{z=1} = \varepsilon\omega\sum_k\frac{C_k}{\sigma_k^2-\omega^2}(\varphi_k^*)_{z=1}\sin\omega t \tag{2.6}$$

It is not difficult to see that each subsequent approximation can be obtained by iterating these calculations.

3. The case of resonance. The solution in the form (2.5) loses meaning if $\omega \to \sigma_k$. In order to study the character of the oscillations in this case, we will consider the "mistuning" of $\sigma_k^2 - \omega^2$ to be small, i.e. $\sigma_k^2 - \omega^2 = \epsilon\mu$. We seek a solution in the form

$$\varphi = \varepsilon^{1/3}\varphi + \varepsilon^{2/3}\varphi_2 + \varepsilon\varphi_3 + \cdots$$

To determine ϕ_i and ζ_i we again have the conditions (2.1) and (2.2) and also the conditions

$$\left(\frac{\partial\varphi_i}{\partial x}\right)_{x=0}=0,\qquad \left(\frac{\partial\varphi_i}{\partial x}\right)_{x=a}=0,\qquad \left(\frac{\partial\varphi_i}{\partial z}\right)_{z=0}=0\qquad (i=1,2)$$

Assuming

$$\varphi_1=\sum_k f_k^{(1)}(t)\,\varphi_k^*$$

we find that

$$\varphi_1=\varphi_k^* A\cos\psi\qquad (\psi=\omega t+\alpha)$$

where the constants A and α are subject to determination. The functions ϕ_2 and ζ_2 are determined analogously; we obtain

$$\varphi_2=(a^{-1/2}d_0+d_{2k}\varphi_{2k}^*)\sin 2\psi+A_1\cos(\omega t+\alpha_1)$$

$$\zeta_2=-2\omega\,[a^{-1/2}d_0+d_{2k}(\varphi_k^*)_{z=1}-A^2\lambda_k\omega^2(\varphi_k^*)_{z=1}^2]\cos 2\psi-$$
$$-\frac{1}{2}A^2(\nabla\varphi_k^*)_{z=1}^2\cos^2\psi+A_1\omega\sin(\omega t+\alpha_1)(\varphi_k^*)_{z=1}$$

$$\left(d_0=\frac{A^2\omega\,(5\lambda_k^2+\lambda_k^{*2})}{8\sqrt{2}\lambda_k a},\quad d_{2k}=\frac{A^2\omega\,(5\lambda_k^2-3\lambda_k^{*2})}{2\sqrt{4a}\,(\lambda_{2k}-4\lambda_k)}\right)$$

Here A_1 and α_1 are new unknown constants. For the third approximation we obtain the condition

$$\left(\frac{\partial^2\varphi_3}{\partial t^2}\right)_{z=1}+\left(\frac{\partial\varphi_3}{\partial z}\right)_{z=1}=L_3(\varphi_1,\varphi_2,\zeta_1,\zeta_2)$$

where

$$L_3=\frac{\partial^2}{\partial t^2}\left[\frac{\partial\varphi_2}{\partial z}\zeta_1+\frac{1}{2}\frac{\partial\varphi_1}{\partial z}\zeta_1^2+\frac{\partial\varphi_1}{\partial z}\zeta_2\right]-\frac{\partial}{\partial t}(\nabla\varphi_1\cdot\nabla\varphi_2)-\qquad (z=1)$$
$$-\frac{1}{2}\frac{\partial}{\partial t}\left[\zeta_1\frac{\partial}{\partial z}(\nabla\varphi_1)^2\right]-\frac{\partial\varphi_1}{\partial z^2}\zeta_2-\frac{1}{2}\frac{\partial^3\varphi_2}{\partial z^3}\zeta_1^2-\frac{\partial^2\varphi_2}{\partial z^2}\zeta_1+\frac{\partial\varphi_2}{\partial x}\frac{\partial\zeta_1}{\partial x}+\frac{\partial\varphi_1}{\partial x}\frac{\partial\zeta_2}{\partial x}+\frac{\partial^2\varphi_1}{\partial x\partial z}\frac{\partial\zeta_1}{\partial x}\zeta_1$$

For ϕ_3 we have the condition on the boundary

$$\left(\frac{\partial\varphi_3}{\partial x}\right)_{x=0}=\omega\cos\omega t,\qquad \left(\frac{\partial\varphi_3}{\partial x}\right)_{x=0}=0,\qquad \left(\frac{\partial\varphi_3}{\partial z}\right)_{z=0}=0$$

We will seek a function ϕ_3 in the form $\phi_3=\phi_{11}\cos\omega t+\phi_{32}$, where ϕ_{11} is determined in Section 2 and ϕ_{32} is harmonic in a region τ satisfying the conditions

$$\left(\frac{\partial\varphi_{32}}{\partial x}\right)_{x=0}=0,\qquad \left(\frac{\partial\varphi_{32}}{\partial x}\right)_{x=a}=0,\qquad \left(\frac{\partial\varphi_{32}}{\partial z}\right)_{z=0}=0$$

Then

$$\left(\frac{\partial^2\varphi_{32}}{\partial t^2}\right)_{z=1}+\left(\frac{\partial\varphi_{32}}{\partial z}\right)_{z=1}=L_3+\left(\frac{\partial\varphi_{11}}{\partial z}\right)_{z=1}\cos\omega t \tag{3.1}$$

Assuming

$$\varphi_{32}=\sum_{0}^{\infty} f_n^{(3)}\varphi_n^*$$

and substituting in (3.1), we obtain

$$\sum_{n=0}^{\infty}\left(\frac{d^2}{dt^2}f_n^{(3)}+\lambda_n f_n^{(3)}\right)(\varphi_n^*)=$$

$$=A^3P(a,k)(\varphi_k^*)_{z=1}(\cos\omega t\cos\alpha-\sin\omega t\sin\alpha)+\cdots+\left(\frac{\partial\varphi_{11}}{\partial z}\right)_{z=1}\cos\omega t \tag{3.2}$$

where

$$P=\left[\frac{\lambda_k}{8a(\lambda_{2k}-4\lambda_k)}\left(\lambda_k^2+\lambda_k\lambda_{2k}+\lambda_k^{*2}\right)(5\lambda_k^2-3\lambda_k^{*2})-\right.$$
$$\left.-\frac{1}{a}\left(\lambda_k^2\lambda_k^{*2}-\frac{11}{16}\lambda_k^4+\frac{13}{16}\lambda_k^{*4}\right)\right]$$

For the existence of the periodic solution (3.2) it is necessary and sufficient that

$$A^3P\cos\alpha+C_k=0,\qquad A^3P\sin\alpha=0$$

where C_k is determined from Formula (2.5). Hence

$$A=C_k^{1/3}P^{-1/3},\qquad \alpha=0$$

and

$$\varphi=\varepsilon^{1/3}\varphi_{11}\cos\omega t+\varepsilon^{1/3}C_k^{1/3}P^{-1/3}\sqrt{\frac{2}{a}}\,\mathrm{ch}\frac{k\pi z}{a}\,\mathrm{sch}\frac{k\pi}{a}\cos\frac{k\pi}{a}\cos\omega t+\cdots$$

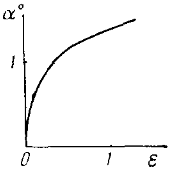

Fig. 2.

Therefore, we obtain a relation between the value of the amplitude of the oscillation of the wave-generator and the value of the amplitude of the resonance oscillation which was sought

$$a^0=\varepsilon^{1/3}\sqrt{\frac{2}{a}}\,\omega C_k^{1/3}=\omega\sqrt[3]{\varepsilon\frac{4\omega th\,k\pi/a}{k\pi a\,P}}$$

Example. Let $a=5\pi$, $k=1$, then (Fig. 2)

$\lambda_1^*=0.2,\quad \lambda_k=0.04088,\quad \lambda_{2k}=0.15196,\quad P=0.00015,\quad \omega=0.202,\quad a^0=0.562\varepsilon^{1/3}$

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Translated by R.D.C.